

COMPACTIFICATIONS OF ω AND THE BANACH SPACE c_0

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ABSTRACT. We investigate for which compactifications $\gamma\omega$ of the discrete space of natural numbers ω , the natural copy of the Banach space c_0 is complemented in $C(\gamma\omega)$. We show, in particular, that the separability of the remainder $\gamma\omega \setminus \omega$ is neither sufficient nor necessary for c_0 being complemented in $C(\gamma\omega)$ (for the latter our result is proved under the continuum hypothesis). We analyse, in this context, compactifications of ω related to embeddings of the measure algebra into $P(\omega)/fin$.

We also prove that a Banach space $C(K)$ contains a rich family of complemented copies of c_0 whenever the compact space K admits only measures of countable Maharam type.

1. INTRODUCTION

If X is a Banach space and Y is a closed subspace of X then Y is said to be complemented in X if there is a closed subspace Z of X such that $X = Y \oplus Z$. This is equivalent to saying that there is a bounded linear operator P from X onto Y which is a projection, i.e. $P \circ P = P$. Recall that typically, unless a Banach space X is isomorphic to a Hilbert space, there are many closed uncomplemented subspaces of X .

The classical Banach space c_0 plays a special role when we speak of complementability: by Sobczyk's theorem [26] every isomorphic copy of c_0 is complemented in any separable superspace. Cabello Sanchez, Castillo and Yost [5] offer an interesting discussion of various proofs and aspects of Sobczyk's theorem; see also a survey paper by Godefroy [17].

Complementability of isomorphic copies of c_0 has been investigated for nonseparable spaces. A Banach space X is said to have the *Sobczyk property* if every subspace of X isomorphic to c_0 is complemented in X . Moltó [22] singled out a certain topological property of the *weak** topology in X^* ensuring that X has the Sobczyk property. Correa and Tausk [9] proved that the space $C(K)$ has the Sobczyk property whenever K is a compact line (generalizing an earlier result from [23], where the same was proved for K being the double arrow space); see also [2], [6], [14], [16] for related results.

Let $\gamma\omega$ be a compactification of the discrete space ω of natural numbers. Then c_0 can be naturally identified with the subspace Y of $C(\gamma\omega)$, where

$$Y = \{f \in C(\gamma\omega) : f|(\gamma\omega \setminus \omega) \equiv 0\},$$

simply by identifying the unit vector e_n in c_0 with $\chi_{\{n\}} \in C(\gamma\omega)$. In the sequel, we shall call the space Y the *natural copy* of c_0 in $C(\gamma\omega)$. We also use the following terminology.

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Definition 1.1. *We say that a compactification $\gamma\omega$ is smooth if the natural copy of c_0 is complemented in $C(\gamma\omega)$.*

The main problem that is considered in the present paper may be stated informally as follows.

Question 1.2. *Which compactifications $\gamma\omega$ are smooth?*

Note that every metrizable compactification $\gamma\omega$ is smooth because $C(\gamma\omega)$ is then separable and every copy of c_0 is complemented in $C(\gamma\omega)$ by Sobczyk's theorem. On the other hand, the maximal compactification $\beta\omega$ is not smooth: $C(\beta\omega)$ is isometric to l_∞ and, by Phillips' theorem [24], c_0 is not complemented in l_∞ . In fact $C(\beta\omega)$ is a Grothendieck space so it contains no complemented copies of c_0 (see the next section). Note also that if we have two comparable compactifications $\gamma_1\omega \leq \gamma_2\omega$, in the sense that there is a continuous surjection $\gamma_2\omega \rightarrow \gamma_1\omega$ that does not move points from ω , then $\gamma_1\omega$ is smooth provided $\gamma_2\omega$ is smooth. Thus smooth compactifications form a natural subclass of all compactifications of ω and 1.2 calls for a reasonable characterization of smoothness.

Question 1.2 has been motivated by Castillo [7] and by conversations with Wiesław Kubiś and Piotr Koszmider. In particular, W. Kubiś observed that if $\gamma\omega$ is smooth then the remainder $\gamma\omega \setminus \omega$ must carry a strictly positive measure (see section 5), and asked if the converse implication holds.

The paper is organized as follows. In section 2 we recall the standard facts related to complementability of c_0 . In section 3 we introduce the terminology and notation concerning Boolean algebras and finitely additive measures and then translate facts from section 2 to the setting that is used throughout the paper.

In section 4 we consider compactifications of ω defined by subalgebras \mathfrak{A} of $P(\omega)$ containing all finite sets and such that the quotient map $\mathfrak{A} \rightarrow \mathfrak{A}/fin$ admits a lifting. We prove in particular that every separable zerodimensional compact space is homeomorphic to the remainder of a smooth compactification (Theorem 4.3).

Our main results read as follows.

- (1) If we take a compactification $\gamma\omega$ related to an embedding of the measure algebra into $P(\omega)/fin$ then $\gamma\omega$ is not smooth (Theorem 5.2). Since in such a case $\gamma\omega \setminus \omega$ is homeomorphic to the Stone space of the measure algebra, $\gamma\omega \setminus \omega$ does carry a strictly positive measure. A related result on such $\gamma\omega$ is a content of Theorem 5.6.
- (2) Under CH there is a smooth compactification with a nonseparable remainder, see Theorem 6.1.
- (3) There is a non-smooth compactification of ω with a separable remainder, see Theorem 7.1.

The conclusion is that, as it seems, the smoothness of $\gamma\omega$ is not directly related to simple topological properties of $\gamma\omega \setminus \omega$. In fact, a smooth compactification may have the same remainder as another non-smooth one, see Corollary 4.4. We do not know if (2) above is provable in the usual set theory; it is likely that for our argument we can relax CH to

the assumption $\mathfrak{b} = \mathfrak{c}$. However, we have not been able to show without additional set-theoretic assumptions a formally weaker assertion: *there is a compactification of ω with a nonseparable remainder that carries a strictly positive measure*, cf. Drygier and Plebanek [12].

In the final section we prove a general result on $C(K)$ spaces containing complemented copies of c_0 . Theorem 8.4 says that if a compact space K has a certain measure theoretic property then every isomorphic copy of c_0 inside $C(K)$ contains a complemented subcopy of c_0 . Our result is related to the work of Moltó [22] and Galego and Plichko [16].

2. PRELIMINARIES

In the sequel, K (possibly with some subscript) always denotes a compact Hausdorff space and $C(K)$ stands for the Banach space of (real-valued) continuous functions on K equipped with the usual supremum norm. The dual space $C(K)^*$ is identified with the space $\mathcal{M}(K)$ of all signed Radon measures of bounded variation defined on the Borel σ -algebra on K . For $\mu \in \mathcal{M}(K)$ and $f \in C(K)$ we write $\mu(g) = \int_K f \, d\mu$ for simplicity. Recall that every $\mu \in \mathcal{M}(K)$ can be written as $\mu = \mu^+ - \mu^-$, where μ^+, μ^- are nonnegative orthogonal measures. Then $|\mu|$, the total variation of μ , is defined as $|\mu| = \mu^+ + \mu^-$, and the norm of μ is $\|\mu\| = |\mu|(K)$. If $x \in K$ then $\delta_x \in \mathcal{M}(K)$ denotes the probability Dirac measure at x . The basic facts on $C(K)$ and $\mathcal{M}(K)$ may be found in Albiac and Kalton [1] or Diestel [10].

The following well-known lemma, establishing a connection between sequences of measures on K and complementability of c_0 in $C(K)$ originates in Veech's proof of Sobczyk's theorem [27]. We write here $\delta(n, k)$ for the Kronecker symbol.

Lemma 2.1. *Let $T: c_0 \rightarrow C(K)$ be an isomorphic embedding and let $Te_n = \varphi_n$. Then the following conditions are equivalent:*

- (i) $T[c_0]$ is complemented in $C(K)$;
- (ii) *there exist bounded sequences $(\mu_n)_n$ and $(\nu_n)_n$ in $\mathcal{M}(K)$ such that*
 - $\nu_n(\varphi_k) = 0$ for every $n, k \in \omega$,
 - $\mu_n(\varphi_k) = \delta(n, k)$ for every $n, k \in \omega$,
 - $\mu_n - \nu_n \rightarrow 0$ in the weak* topology.

Proof. To check (ii) \Rightarrow (i) define $P: C(K) \rightarrow C(K)$ by

$$Pf = \sum_{n \in \omega} (\mu_n - \nu_n)(f) \cdot \varphi_n.$$

Then P is easily seen to be a bounded projection from $C(K)$ onto $T[c_0]$.

For the converse implication consider the dual operator $T^*: \mathcal{M}(K) \rightarrow c_0^* = l_1$. Since T is an isomorphic embedding, T^* is a surjection so for each $e_n^* = e_n \in l_1$ there exists a measure $\mu_n \in \mathcal{M}(K)$ such that $T^*\mu_n = e_n^*$, and the sequence of μ_n is norm bounded. We have $T^*(\mu_n)(e_k) = e_n^*(e_k) = \delta(n, k)$ and $T^*(\mu_n)(e_k) = \mu_n(Te_k) = \mu_n(\varphi_k)$, so $\mu_n(\varphi_k) = \delta(n, k)$. For every n define a measure $\nu_n \in \mathcal{M}(K)$ putting $\nu_n(f) = \mu_n(f) - \mu_n(Pf)$ for $f \in C(K)$.

Then ν_n vanishes on $P[C(K)]$ and for every $f \in C(K)$, taking $x \in c_0$ such that $Tx = Pf$, we get

$$\mu_n(f) - \nu_n(f) = \mu_n(Pf) = \mu_n(Tx) = e_n^*(x) \rightarrow 0,$$

as required. \square

Here is the most obvious illustration of the previous lemma.

Corollary 2.2. *If K contains a non-trivial converging sequence then $C(K)$ contains a complemented copy of c_0 .*

Proof. Let $(x_n)_n$ be a sequence in K converging to $x \in K$ and such that $x_n \neq x$ for every n . Then it is easy to construct a pairwise disjoint family $\{U_n : n \in \omega\}$ of open subsets of K such that $x_n \in U_n$ for each $n \in \omega$.

For every n we can find a continuous function $f_n : K \rightarrow [0, 1]$ such that $f_n(x_n) = 1$ and f_n vanishes outside U_n . Now if we define $T : c_0 \rightarrow C(K)$ by $Te_n = f_n$ then $T[c_0]$ is complemented in $C(K)$. Indeed, we can apply Lemma 2.1 with $\mu_n = \delta_{x_n}$ and $\nu_n = \delta_x$ for every n . \square

The way we stated Lemma 2.1 is motivated by its application to compactifications of ω .

Corollary 2.3. *A compactification $\gamma\omega$ is smooth if and only if there exists a bounded sequence of measures $(\nu_n)_n$ in $\mathcal{M}(\gamma\omega)$ such that $|\nu_n|(\omega) = 0$ for every n and $\nu_n - \delta_n \rightarrow 0$ in the weak* topology.*

Let us note that the smoothness of $\gamma\omega$ is directly related to the existence of a certain extension operator. If L is a closed subspace of a compact space K then an extension operator $E : C(L) \rightarrow C(K)$ is a bounded linear operator such that $E(f)|_L = f$ for every $f \in C(L)$; see Avilés and Marciszewski [3] for a recent result on extension operators and references therein.

Lemma 2.4. *A compactification $\gamma\omega$ is smooth if and only if there is an extension operator $C(\gamma\omega \setminus \omega) \rightarrow C(\gamma\omega)$.*

Proof. Suppose that $P : C(\gamma\omega) \rightarrow c_0$ is a bounded projection (where c_0 is identified with its natural copy inside $C(\gamma\omega)$). For $f \in C(\gamma\omega \setminus \omega)$ take any extension of f to a function $g \in C(\gamma\omega)$ and define $E(f) = g - Pg$. Note that $E(f)$ is uniquely defined: if $g' \in C(\gamma\omega)$ is another extension of f then $g' - g$ vanishes on the remainder so $g' - g \in c_0$ and $P(g' - g) = g' - g$, that is $g' - Pg' = g - Pg$.

Suppose now that $E : C(\gamma\omega \setminus \omega) \rightarrow C(\gamma\omega)$ is an extension operator. Then $Pg = g - E(g|_{\gamma\omega \setminus \omega})$ defines a projection from $C(\gamma\omega)$ onto c_0 . \square

We shall now recall the notion of a Grothendieck space which for $C(K)$ spaces means being anti-Sobczyk.

Definition 2.5. *A Banach space is said to be a Grothendieck space if every weak* null sequence $(x_n^*)_n$ in X^* converges weakly (i.e. $x^{**}(x_n^*) \rightarrow 0$ for every $x^{**} \in X^{**}$).*

For the proof of the following see Cembranos [8].

Theorem 2.6. *Given a compact space K , the space $C(K)$ is Grothendieck if and only if $C(K)$ does not contain a complemented copy of c_0 .*

Recall that typical examples of Grothendieck spaces are l_∞ and, more generally, $C(K)$ spaces where K is extremally disconnected compact space, see [1] or [10].

3. BOOLEAN ALGEBRAS AND COMPACTIFICATIONS

We shall consider mainly zerodimensional compactifications of ω and those are naturally related to Boolean subalgebras of $P(\omega)$. If \mathfrak{A} is any Boolean algebra then $\text{ult}(\mathfrak{A})$ denotes its Stone space of all ultrafilters on \mathfrak{A} . If $a \in \mathfrak{A}$ then \hat{a} is the corresponding clopen set in $\text{ult}(\mathfrak{A})$, that is

$$\hat{a} = \{x \in \text{ult}(\mathfrak{A}) : a \in x\}.$$

A family $\mathcal{U} \subseteq p$ is a base of the ultrafilter p if every $A \in p$ contains some $U \in \mathcal{U}$, in other words, if $\{\hat{U} : U \in \mathcal{U}\}$ is a local base at $p \in \text{ult}(\mathfrak{A})$.

Let \mathfrak{A} be any Boolean algebra. Then $\text{ba}(\mathfrak{A})$ will stand for the family of all finitely additive measures μ on \mathfrak{A} that have bounded variation and $\text{ba}_+(\mathfrak{A})$ are finitely additive nonnegative functions. We call any $\mu \in \text{ba}(\mathfrak{A})$ simply a measure. We now recall the following standard facts, see e.g. Semadeni [25], 18.7. Every $\mu \in \text{ba}_+(\mathfrak{A})$ can be transferred onto the algebra of clopen subsets of $\text{ult}(\mathfrak{A})$ by the formula $\hat{\mu}(\hat{a}) = \mu(a)$, and then uniquely extended to a Radon measure on $\text{ult}(\mathfrak{A})$ (that Radon measure is still denoted by $\hat{\mu}$). Note that the *weak** topology on a bounded subset of $\mathcal{M}(\text{ult}(\mathfrak{A}))$ may be seen as the topology of convergence of clopen subsets of $\text{ult}(\mathfrak{A})$. Hence for a bounded sequence μ_n in $\text{ba}(\mathfrak{A})$ we have $\hat{\mu}_n \rightarrow 0$ in the *weak** topology of $\mathcal{M}(\text{ult}(\mathfrak{A}))$ if and only if $\mu_n(a) \rightarrow 0$ for every $a \in \mathfrak{A}$.

If \mathfrak{A} is a subalgebra of $P(\omega)$ and \mathfrak{A} contains *fin*, the family of all finite subsets of ω , then $\text{ult}(\mathfrak{A})$ is a compactification of ω — we simply identify points in ω with principal ultrafilters on \mathfrak{A} . We shall denote such a compactification of ω by $K_{\mathfrak{A}}$ and $K_{\mathfrak{A}}^* = K_{\mathfrak{A}} \setminus \omega$ will stand for its remainder. Note that $K_{\mathfrak{A}}^*$ may be identified with $\text{ult}(\mathfrak{A}/\text{fin})$.

Using the terminology and notation introduced above we can rewrite Corollary 2.3 as follows.

Lemma 3.1. *Let \mathfrak{A} be an algebra such that $\text{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$. Then the compactification $K_{\mathfrak{A}}$ of ω is smooth if and only if there exists a bounded sequence $(\nu_n)_n$ in $\text{ba}(\mathfrak{A})$ such that*

- (i) $\nu_n|_{\text{fin}} \equiv 0$ for every n , and
- (ii) $\nu_n - \delta_n \rightarrow 0$ on \mathfrak{A} , that is $(\nu_n - \delta_n)(A) \rightarrow 0$ for every $A \in \mathfrak{A}$.

Note that in case $A \in \mathfrak{A}$ is infinite and co-infinite, condition (ii) above is equivalent to

$$(3.1.1) \quad \lim_{n \in A} \nu_n(A) = 1, \quad \lim_{n \notin A} \nu_n(A) = 0.$$

We shall below often enlarge a given algebra $\mathfrak{A} \subseteq P(\omega)$ by adding a new set $X \subseteq \omega$; let $\mathfrak{A}[X]$ be the algebra generated by \mathfrak{A} and X . Then

$$\mathfrak{A}[X] = \{(A \cap X) \cup (A' \setminus X) : A, A' \in \mathfrak{A}\}.$$

If $\mu \in \text{ba}_+(\mathfrak{A})$ then μ_* and μ^* denote the corresponding inner and outer measure defined as

$$\mu_*(X) = \sup\{\mu(A) : A \in \mathfrak{A}, A \subseteq X\}, \quad \mu^*(X) = \inf\{\mu(A) : A \in \mathfrak{A}, A \supseteq X\}.$$

The following fact on extensions of finitely additive measures is due to Łoś and Marczewski [21].

Theorem 3.2. *Given an algebra \mathfrak{A} , $\mu \in \text{ba}_+(\mathfrak{A})$ and any X the following formulas define extensions of μ to $\mu_1, \mu_2 \in \text{ba}_+(\mathfrak{A}[X])$*

$$\mu_1((A \cap X) \cup (A' \setminus X)) = \mu_*(A \cap X) + \mu^*(A' \setminus X),$$

$$\mu_2((A \cap X) \cup (A' \setminus X)) = \mu^*(A \cap X) + \mu_*(A' \setminus X).$$

Consequently, for every t satisfying $\mu_*(X) \leq t \leq \mu^*(X)$, there is an extension of μ to $\mu_t \in \text{ba}_+(\mathfrak{A}[X])$ such that $\mu_t(X) = t$.

The following lemma on convergence of extended probability measures will be needed in section 6.

Lemma 3.3. *Let \mathfrak{A} be an algebra such that $\text{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$ and let $(\nu_n)_n$ be a sequence of probability measures from $\text{ba}_+(\mathfrak{A})$ such that $\nu_n - \delta_n \rightarrow 0$. Further let, for every n , $\tilde{\nu}_n \in \text{ba}_+(\mathfrak{A}[X])$ be any extension of ν_n .*

If $\tilde{\nu}_n(X) - \delta_n(X) \rightarrow 0$ then $\tilde{\nu}_n - \delta_n \rightarrow 0$ on $\mathfrak{A}[X]$.

Proof. We use here 3.1.1. For any $A \in \mathfrak{A}$ if n runs through $A \cap X$ then $\tilde{\nu}_n(X) \rightarrow 1$ and $\tilde{\nu}_n(A) = \nu_n(A) \rightarrow 1$ so $\tilde{\nu}_n(A \cap X) \rightarrow 1$ (using $\tilde{\nu}_n(\omega) = \nu_n(\omega) = 1$).

Take any $\varepsilon > 0$. Then $\nu_n(A) < \varepsilon$ if $n \notin A$ and $n \geq n_0$ and $\tilde{\nu}_n(X) < \varepsilon$ whenever $n \geq n_1$ and $n \notin X$. Hence for $n \geq \max(n_0, n_1)$, if $n \notin A \cap X$ then either $n \notin A$ and $\tilde{\nu}_n(A \cap X) \leq \nu_n(A) < \varepsilon$ or $n \notin X$ and $\tilde{\nu}_n(A \cap X) \leq \tilde{\nu}_n(X) < \varepsilon$.

The convergence of $\tilde{\nu}_n(A \setminus X)$ may be checked in a similar way. \square

Recall that a nonnegative measure $\mu \in \text{ba}(\mathfrak{A})$ is said to be *nonatomic* if for every $\varepsilon > 0$ there is a finite partition of $1_{\mathfrak{A}}$ into pieces of measure $< \varepsilon$. A signed measure μ is nonatomic if its variation $|\mu|$ is nonatomic. We shall use the following two simple observations.

Lemma 3.4. *Given a Boolean algebra \mathfrak{A} and a signed measure μ on \mathfrak{A} , μ is nonatomic if and only if $\inf\{|\mu|(A) : A \in p\} = 0$ for every $p \in \text{ult}(\mathfrak{A})$, and this is equivalent to saying that $\hat{\mu}$ is a Radon measure on $K_{\mathfrak{A}}$ vanishing on points of $K_{\mathfrak{A}}$.*

If μ is nonatomic on \mathfrak{A} , $a \in \mathfrak{A}$ and $t < \mu(a)$ then for every $\varepsilon > 0$ there is $b \in \mathfrak{A}$ such that $b \leq a$ and $|\mu(b) - t| < \varepsilon$.

4. LIFTINGS

Let \mathfrak{A} be an algebra such that $\text{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$. Consider the canonical quotient map

$$\mathfrak{A} \rightarrow \mathfrak{A}/\text{fin}, \mathfrak{A} \ni A \rightarrow A^\bullet \in \mathfrak{A}/\text{fin}.$$

By a *lifting* of the quotient map we mean a Boolean homomorphism $\rho: \mathfrak{A}/\text{fin} \rightarrow \mathfrak{A}$ such that $\rho(a)^\bullet = a$ for every $a \in \mathfrak{A}/\text{fin}$.

Lemma 4.1. *For an algebra \mathfrak{A} such that $\text{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$, the quotient map $\mathfrak{A} \rightarrow \mathfrak{A}/\text{fin}$ admits a lifting if and only there exists an Boolean algebra $\mathfrak{B} \subseteq P(\omega)$ such that every $B \in \mathfrak{B} \setminus \{\emptyset\}$ is infinite and \mathfrak{A} is equal to $\text{alg}(\mathfrak{B} \cup \text{fin})$, the algebra generated by \mathfrak{B} and fin .*

Proof. If ρ is a lifting, then put $\mathfrak{B} = \rho[\mathfrak{A}/\text{fin}]$. For every nonzero $a \in \mathfrak{A}/\text{fin}$ the set $\rho(a)$ is infinite because $\rho(a)^\bullet = a \neq 0$.

If $\mathfrak{A} = \text{alg}(\mathfrak{B} \cup \text{fin})$ then, by the property of \mathfrak{B} , for any element $a \in \mathfrak{A}/\text{fin}$ there exists exactly one $B_a \in \mathfrak{B}$ such that $B_a^\bullet = a$. Therefore we can define $\rho(a) = B_a$ and ρ is a homomorphism. \square

Theorem 4.2. *If $\text{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$ is such an algebra that the quotient map $\mathfrak{A} \rightarrow \mathfrak{A}/\text{fin}$ admits a lifting then the compactification $K_{\mathfrak{A}}$ of ω is smooth.*

Proof. By Lemma 4.1 there exists an algebra \mathfrak{A} of infinite subsets of ω such that $\mathfrak{A} = \text{alg}(\mathfrak{B} \cup \text{fin})$. For every n consider the ultrafilter $p_n = \{B \in \mathfrak{B} : n \in B\}$ on \mathfrak{B} . Then p_n extends to the nonprincipal ultrafilter x_n on \mathfrak{A} , where

$$x_n = \{B \triangle I : B \in p_n, I \in \text{fin}\}.$$

It follows that $\delta_{x_n} - \delta_n \rightarrow 0$ on \mathfrak{A} since for every $A \in \mathfrak{A}$ we have $\delta_{x_n}(A) - \delta_n(A) = 0$ except for finitely many n 's. Thus $K_{\mathfrak{A}}$ is smooth by Lemma 3.1. \square

We note that thanks to Theorem 4.2 one can easily define relatively big smooth compactifications of ω . Take for instance an independent sequence $\{B_\alpha : \alpha < \mathfrak{c}\}$ in $P(\omega)$ such that the set $\bigcap_{\alpha \in I} B_\alpha^{\varepsilon(\alpha)}$ is infinite for every finite $I \subseteq \mathfrak{c}$ and every $\varepsilon : I \rightarrow \{0, 1\}$. Then the algebra \mathfrak{A} generated by all B_α 's and fin is such that \mathfrak{A}/fin has a lifting by Lemma 4.1 and, by Theorem 4.2, $K_{\mathfrak{A}}$ is a smooth compactification of ω which remainder is homeomorphic to the Cantor cube $2^\mathfrak{c}$. This can be generalized as follows.

Theorem 4.3. *If L is a separable zerodimensional compact space then there exists a smooth compactification $\gamma\omega$ such that $\gamma\omega \setminus \omega$ is homeomorphic to L .*

Proof. We write $\text{Clopen}(L)$ for the algebra of clopen subsets of L . We define an embedding $\varphi : \text{Clopen}(L) \rightarrow P(\omega)$ as follows. Take a partition $\{B_d : d \in D\}$ of ω into infinite sets and let for $V \in \text{Clopen}(L)$

$$\varphi(V) = \bigcup_{d \in V \cap D} B_d.$$

Then for $U, V \in \text{Clopen}(L)$, if $\varphi(U) = \varphi(V)$ then $U \cap D = V \cap D$ so $U = V$. It is easy to check that φ is indeed an isomorphic embedding of $\text{Clopen}(L)$ into $P(\omega)$ and the algebra $\mathcal{B} = \varphi[\text{Clopen}(L)]$ has the property that every nonempty $B \in \mathfrak{B}$ is infinite.

Letting \mathfrak{A} be the algebra in $P(\omega)$ generated by \mathfrak{B} and fin we conclude from Theorem 4.2 that $K_{\mathfrak{A}}$ is a smooth compactification. Moreover, $K_{\mathfrak{A}}^*$ can be identified with $\text{ult}(\mathfrak{B})$ which is homeomorphic to $\text{ult}(\text{Clopen}(L))$, so to L itself. \square

We prove below that there is a compactification $\gamma\omega$ which is not smooth but $\gamma\omega \setminus \omega$ is separable (and zerodimensional). The conclusion is that smoothness of a compactification $\gamma\omega$ cannot be decided by examining topological properties of $\gamma\omega \setminus \omega$ alone.

Corollary 4.4. *There are two compactifications $\gamma\omega$ and $\eta\omega$ such that $\gamma\omega \setminus \omega \cong \eta\omega \setminus \omega$, while $\eta\gamma$ is smooth and $\gamma\omega$ is not smooth.*

Proof. Take $\gamma\omega$ as in Theorem 7.1, that is a non-smooth zerodimensional compactification with $L = \gamma\omega \setminus \omega$ separable. By Theorem 4.3 there is a smooth compactification $\eta\omega$ such that $\eta\omega \setminus \omega \cong L \cong \gamma\omega \setminus \omega$. \square

We finish the section by the following side remark. If $\text{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$ is an algebra and the quotient map $\mathfrak{A} \rightarrow \mathfrak{A}/\text{fin}$ admits a lifting then the algebra \mathfrak{A}/fin is σ -centred. Note the reverse implication does not hold: If we take \mathfrak{A} as in Corollary 4.4 then $K_{\mathfrak{A}}$ is not smooth so \mathfrak{A} does not have a lifting by Theorem 4.2. On the other hand, \mathfrak{A}/fin is σ -centred since it is isomorphic to the clopen algebra of a separable space $K_{\mathfrak{A}}^*$.

5. THE MEASURE ALGEBRA

We start this section by the following observation due to W. Kubiś.

Theorem 5.1. *If $\gamma\omega$ is a smooth compactification then its remainder carries a strictly positive measure.*

Proof. Take a sequence $(\nu_n)_n$ as in Lemma 2.3 and consider $\mu = \sum_n 2^{-n} |\nu_n|$. Then μ is a finite nonnegative measure on $\gamma\omega$ and $\mu(\omega) = 0$. Let $U \subseteq \gamma\omega$ be an open set such that $U \cap (\gamma\omega \setminus \omega) \neq \emptyset$. Take a continuous function $g: \gamma\omega \rightarrow [0, 1]$ that vanishes outside U and $g(x_0) = 1$ for some $x_0 \in U$. Then the set $V = \{g > 1/2\}$ contains infinitely many $n \in \omega$. Since $\nu_n(g) - g(n) \rightarrow 0$ we conclude that $\nu_n(g) > 0$ for some n and this gives

$$\mu(U) \geq \mu(g) \geq 2^{-n} |\nu_n|(g) > 0,$$

so the measure μ is positive on every nonempty open subset of $\gamma\omega \setminus \omega$. \square

We show in this section that under CH there are non-smooth compactifications $\gamma\omega$ such that $\gamma\omega \setminus \omega$ carries a strictly positive nonatomic measure.

We consider here the measure algebra \mathfrak{M} , that is the quotient $\text{Bor}[0, 1]/\mathcal{N}$, where \mathcal{N} is the ideal of Lebesgue null sets. We denote by λ the measure on \mathfrak{M} defined from the Lebesgue measure. We write $S = \text{ult}(\mathfrak{M})$ for its Stone space. It is well-known that the space $C(S)$ is isometric to $L_\infty[0, 1]$.

By the classical Parovicenko theorem, under CH there is an isomorphic embedding $\varphi: \mathfrak{M} \rightarrow P(\omega)/fin$. Define an algebra $\mathfrak{A} \subseteq P(\omega)$ as the algebra of all finite modifications of elements of $\varphi[\mathfrak{M}]$ and consider the Stone space $K_{\mathfrak{A}}$. Then $K_{\mathfrak{A}}$ is a compactification of ω such that $K_{\mathfrak{A}}^*$ is homeomorphic to S .

We shall investigate the space $K_{\mathfrak{A}}$ in Theorem 5.4. Note first that $C(K_{\mathfrak{A}}^*) = C(S)$ is a Grothendieck space so it contains no complemented copy of c_0 .

Lemma 5.2. *For every family $\{A_n: n \in \omega\} \subseteq \mathfrak{A}$ there is $B \in \mathfrak{A}$ which is an almost upper bound of that family, in the sense that $A_n \subseteq^* B$ for every n and $B \cap A$ is finite whenever $A \in \mathfrak{A}$ is such that $A \cap A_n = \emptyset$ for every n .*

Proof. For every n take $a_n \in \mathfrak{M}$ such that $\varphi(a_n) = A_n^\bullet$. The algebra \mathfrak{M} is complete so the family $\{a_n: n \in \omega\}$ has the least upper bound $b \in \mathfrak{M}$. Now $B \in \mathfrak{A}$ such that $B^\bullet = \varphi(b)$ is as required. \square

Theorem 5.3. *Let $fin \subseteq \mathfrak{A} \subseteq P(\omega)$ be an algebra such that $K_{\mathfrak{A}}^* \cong S$. Then $K_{\mathfrak{A}}$ is a compactification of ω that is not smooth.*

Proof. By Lemma 3.1 it is enough to check that whenever $(\nu_n)_n$ is a bounded sequence, where every $\nu_n \in \text{ba}(\mathfrak{A})$ vanishes on finite sets, then $\nu_n - \delta_n$ does not converge to 0. Suppose otherwise; let $\nu_n(A) - \delta_n(A) \rightarrow 0$ for every $A \in \mathfrak{A}$.

Note first that there is an infinite $T \in \mathfrak{A}$ such that every ν_n is nonatomic on the algebra $\mathfrak{A}_T = \{A \in \mathfrak{A}: A \subseteq T\}$ of subsets of T . Indeed, every $\widehat{\nu}_n$, as a Radon measure on $K_{\mathfrak{A}}$ is concentrated on S and the set $\{x \in S: \widehat{\nu}_n(\{x\}) \neq 0\}$ is at most countable (since $|\widehat{\nu}_n|(S)$ is finite; see Lemma 3.4). The space S is not separable and therefore there is a nonzero $a \in \mathfrak{M}$ such that $\widehat{\nu}_n(\{x\}) = 0$ for every n and every $x \in \widehat{a}$. Take $T \in \mathfrak{A}$ such that $T^\bullet = a$; then T is as required.

Fix $\varepsilon = 1/8$. Take any $A_1 \in \mathfrak{A}$ such that $A_1 \subseteq T$ and $A_1, T \setminus A_1$ are infinite. Since for every $n \in A_1$ we have $\delta_n(A_1) = 1$, so $\lim_{n \in A_1} \nu_n(A_1) = 1$. Hence there exists $n_1 \in A_1$ such that $\nu_{n_1}(A_1) \geq 1 - \varepsilon$. Moreover, since the variation of ν_{n_1} is finite, there exists infinite $D_1 \in \mathfrak{A}$ such that $D_1 \subseteq T \setminus A_1$ and $|\nu_{n_1}|(D_1) < \varepsilon$.

In a similar way for every $k > 1$ there exist infinite sets $A_k, D_k \in \mathfrak{A}$ and $n_1 < n_2 < \dots$ such that

- (a) $A_k \subseteq D_{k-1}$ and $D_k \subseteq D_{k-1} \setminus A_k$;
- (b) $D_{k-1} \setminus A_k$ is infinite;
- (c) $n_k \in A_k$;
- (d) $\nu_{n_k}(A_k) \geq 1 - \varepsilon$ and $|\nu_{n_k}|(D_k) < \varepsilon$.

Since all the measures ν_n are nonatomic on T , by Lemma 3.4 for every $k \in \omega$ there exists a set $B_k \in \mathfrak{A}$ such that $B_k \subseteq A_k$ and $|\nu_{n_k}(B_k) - \frac{1}{2}| < \varepsilon$.

As A_k are pairwise disjoint and $n_k \in A_k$, it follows from $\nu_n - \delta_n \rightarrow 0$ that there is an infinite $N \subseteq \omega$ such that for every $k \in N$

$$(5.3.1) \quad \left| \nu_{n_k} \left(\bigcup_{j < k, j \in N} B_j \right) \right| < \varepsilon.$$

Let $B \in \mathfrak{A}$ be an almost upper bound for $\{B_k : k \in N\}$ as in Lemma 5.2. Write, for simplicity, $B_{<k} = \bigcup_{j < k, j \in N} B_j$, and let $B_{\leq k}$ be defined accordingly. For any $k \in N$ we have

$$\nu_{n_k}(B) = \nu_{n_k}(B_{<k}) + \nu_{n_k}(B_k) + \nu_{n_k}(B \setminus B_{\leq k}),$$

where $B \setminus B_{\leq k} \subseteq^* D_k$. Using 5.3.1, condition (d) and the fact that ν_{n_k} vanishes on finite sets we get

$$\left| \nu_{n_k}(B) - \frac{1}{2} \right| < 3\varepsilon, \quad \text{so } 1/8 < \nu_{n_k}(B) < 7/8,$$

for every $k \in N$ (note that $1/2 + 3\varepsilon = 1/2 + 3/8 = 7/8$).

On the other hand, consider the set $J = \{k \in N : n_k \in B\}$. If J is infinite then $\nu_{n_k}(B) \rightarrow 1$ for $k \in J$. If $N \setminus J$ is infinite we should have $\nu_{n_k}(B) \rightarrow 0$ for $k \in N \setminus J$, and in both cases this is a contradiction. \square

We shall now prove that for \mathfrak{A} as in Theorem 5.3 the space $C(K_{\mathfrak{A}})$ need not to be Grothendieck. Consider the family \mathfrak{D} of all subsets $A \subseteq \omega$ having the asymptotic density

$$d(A) = \lim_n |A \cap n|/n.$$

Observe that the asymptotic density does not depend on finite modifications of the set $A \subseteq \mathfrak{D}$, so we can also treat d as the function defined on \mathfrak{D}/fin .

In the proof of Theorem 5.6 we shall use the following result of Frankiewicz and Gutek and a simple remark 5.5.

Theorem 5.4 ([15]). *Assuming CH there exists an embedding $\varphi: \mathfrak{M} \rightarrow P(\omega)/\text{fin}$ such that $\varphi(a) \in \mathfrak{D}/\text{fin}$ and $d(\varphi(a)) = \lambda(a)$ for every $a \in \mathfrak{M}$.*

Lemma 5.5. *There exists a family $\{I_n\}_{n \in \omega}$ of pairwise disjoint finite subsets of ω such that $d(A) = \lim_n \frac{|A \cap I_n|}{|I_n|}$ for any $A \in \mathfrak{D}$.*

Proof. Take any increasing sequence of integers k_n such that $\lim_n k_n/k_{n+1} = 0$ and put $I_n = \{i \in \omega : k_n \leq i < k_{n+1}\}$. Then I_n are as required by standard calculations. \square

Theorem 5.6. *Under CH there is a compactification $\gamma\omega$ such that $\gamma\omega \setminus \omega \cong S$ and $C(\gamma\omega)$ contains a complemented copy of c_0 .*

Proof. We use an embedding $\varphi: \mathfrak{M} \rightarrow P(\omega)/\text{fin}$ as in 5.4 and consider \mathfrak{A} as in the beginning of this section and in Theorem 5.3. Take a family of pairwise disjoint intervals I_n as in Lemma 5.5. For every $n \in \omega$ let $f_n = \chi_{I_n} \in C(K_{\mathfrak{A}})$ be the characteristic function of the set I_n .

Let us consider the space $Y = \overline{\text{lin}\langle f_n : n \in \omega \rangle}$. Then Y is a closed subspace of $C(K_{\mathfrak{A}})$ isomorphic to c_0 ; the isomorphism between them is defined by setting $e_n \mapsto f_n$.

Now Y is complemented in $C(K_{\mathfrak{A}})$; indeed, consider the measures

$$\mu_n = \frac{1}{|I_n|} \sum_{i \in I_n} \delta_i \in \text{ba}_+(\mathfrak{A}).$$

Then $\mu_n(f_k)$ equals 1 if $k = n$ and 0 if $n \neq k$ because I_n are pairwise disjoint. Moreover, by Lemma 5.5, $\mu_n \rightarrow d$ on the algebra \mathfrak{A} so we can apply Lemma 2.1. \square

6. LARGE SMOOTH COMPACTIFICATION

Theorem 6.1. *Under CH there exists a smooth compactification $\gamma\omega$ of ω such that $\gamma\omega \setminus \omega$ is not separable.*

This section is devoted to proving Theorem 6.1. The desired compactification will be defined as $K_{\mathfrak{A}}$, where $\mathfrak{A} = \bigcup_{\alpha < \omega_1} \mathfrak{A}_\alpha$, and countable algebras \mathfrak{A}_α are defined inductively. Lemma 6.2 describes the starting point of the construction; we only sketch its proof here since it closely follows the proof of Sobczyk's theorem in Diestel [10]. Subsequent Lemma 6.3 contains the essence of the inductive step

Lemma 6.2. *There exist a countable nonatomic Boolean algebra $\mathfrak{C} \subseteq P(\omega)$ and a sequence of nonatomic probability measures $(\nu_n)_n$ on $\mathfrak{B} = \text{alg}(\mathfrak{C} \cup \text{fin})$ such that $\nu_n|_{\text{fin}} \equiv 0$ for every n and $\nu_n - \delta_n \rightarrow 0$ on \mathfrak{B} .*

Proof. There is an algebra $\mathfrak{C} \subseteq P(\omega)$ isomorphic to $\text{Clopen}(2^\omega)$, the algebra of clopen subsets of the Cantor set. We can copy the standard product measure ν on 2^ω onto \mathfrak{C} (and denote it by the same letter). Put $\mathfrak{B} = \text{alg}(\mathfrak{C} \cup \text{fin})$. Then we have a probability measure μ on \mathfrak{B} defined by $\mu(C \triangle I) = \nu(C)$ for every $C \in \mathfrak{C}$ and $I \in \text{fin}$.

Since \mathfrak{B} is countable the space $K_{\mathfrak{B}}$ is metrizable, so $C(K_{\mathfrak{B}})$ is a separable Banach space. Hence the unit ball $\mathcal{M}_1(K_{\mathfrak{B}})$ is metrizable in its weak^* topology. Let the metric ρ metrize $\mathcal{M}_1(K_{\mathfrak{B}})$. Denote by \mathcal{P} the space of probability measures on $K_{\mathfrak{B}}$ that vanish on ω .

It is not difficult to check that the set of nonatomic $\nu \in \mathcal{P}$ is weak^* dense in \mathcal{P} : consider for instance the convex hull of the family $\mu_C \in \text{ba}_+(\mathfrak{B})$, where $\mu_C(B) = (1/\mu(C)) \cdot \mu(C \cap B)$ and $C \in \mathfrak{C}$. Thus for every n we can choose nonatomic $\nu_n \in \mathcal{P}$ such that

$$\rho(\nu_n, \delta_n) \leq 2 \cdot \text{dist}(\delta_n, \mathcal{P}).$$

Then $\nu_n - \delta_n \rightarrow 0$ in the weak^* topology. \square

Lemma 6.3. *Let $\mathfrak{B} \subseteq P(\omega)$ be a countable Boolean algebra containing fin . Moreover, suppose that:*

- (i) *we are given a set $\{p_j : j \in \omega\}$ of ultrafilters which dense in $K_{\mathfrak{B}}^*$;*
- (ii) *$(\nu_n)_{n \in \omega}$ is a sequence of nonatomic probability measures on \mathfrak{B} ;*
- (iii) *$\nu_n(I) = 0$ for every $I \in \text{fin}$ and every n ;*
- (iv) *$\nu_n - \delta_n \rightarrow 0$ on \mathfrak{B} .*

Then there exists an infinite set $X \subseteq \omega$ such that

- for any extension \tilde{p}_j of p_j to an ultrafilter on $\mathfrak{B}[X]$ the set $\{\tilde{p}_j : j \in \omega\}$ is not dense in $K_{\mathfrak{B}[X]}^*$;
- we can extend every ν_n to a probability measure $\tilde{\nu}_n$ on $\mathfrak{B}[X]$ so that $\tilde{\nu}_n - \delta_n \rightarrow 0$ on $\mathfrak{B}[X]$.

Proof. Since \mathfrak{B} is countable, we fix an enumeration $\mathfrak{B} = \{B_0, B_1, \dots\}$.

CLAIM. There are infinite sets $A_j \in \mathfrak{B}$ and $c(j) \in \omega$ for $j \in \omega$ such that

- (1) $\nu_0(A_j) < 2^{-(j+2)}$ for all $j \in \omega$;
- (2) $A_j \in p_j$;
- (3) for every i and $n \in \omega \setminus A_j$ we have $\nu_n(A_j) < 2^{-(j+2)}$;
- (4) $c(j) \notin A_k$ for all j, k ;
- (5) for every j either $B_j \subseteq \bigcup_{k \leq j} A_k$ or $c(j) \in B_j$.

Proof of Claim. We proceed by induction on j . Suppose that we have already constructed A_0, \dots, A_j and $c(0), \dots, c(j)$ and put $m = \max_{i \leq j} c(i) + 1$ (of course we can additionally assume that $0 \notin A_i$).

Since all the measures ν_i are nonatomic and p_{j+1} is a nonprincipal ultrafilter, there is $A \in p_{j+1}$ such that $A \cap m = \emptyset$ and $\nu_i(A) < 2^{-(j+3)}$ for every $i \leq m$. Since $\nu_n(A) - \delta_n(A) \rightarrow 0$, the set $F = \{n \notin A : \nu_n(A) \geq 2^{-(j+3)}\}$ is finite. Define $A_{j+1} = A \cup F$. Since $\nu_i(F) = 0$ for every i , (1) and (3) are granted by the choice of A . Condition (4) holds since $F \cap m = \emptyset$.

Now we can set $c(j+1) = 0$ or choose $c(j+1) \in B_{j+1} \setminus \bigcup_{k \leq j+1} A_k$, if possible. This verifies Claim.

We take the sets A_j from Claim and prove that the set $X = \bigcup_j A_j$ is as desired. We first check the following properties of X .

- (a) For any $B \in \mathfrak{B}$ if $B \subseteq X$ then $B \subseteq \bigcup_{j \leq N} A_j$ for some $N \in \omega$.
- (b) $\omega \setminus X$ is infinite.
- (c) $\nu_n^*(X) = 1$ for every n .
- (d) $\lim_{n \notin X} (\nu_n)_*(X) = 0$.

Ad (a). Take any $B_j \in \mathfrak{B}$ such that $B_j \subseteq X$. Suppose that $B_j \not\subseteq \bigcup_{i \leq j} A_i$. Then $c(j) \in B_j \setminus X$ by (4) and (5), a contradiction.

Ad (b). Suppose $\omega \setminus X$ is finite. Then, since every finite set belongs to the algebra \mathfrak{B} , $X \in \mathfrak{B}$. By (a) $X \subseteq \bigcup_{i \leq N} A_i$ for some N . Since ν_0 is insensitive to finite modifications of sets, by condition (1)

$$\nu_0(\omega) = \nu_0(X) \leq \nu_0\left(\bigcup_{i \leq N} A_i\right) \leq \sum_{i \leq N} 2^{-(i+2)} \leq 1/2,$$

which contradicts the fact that ν_0 is a probability measure on ω .

Ad (c). For any n , if $B \in \mathfrak{B}$ and $\nu_n(B) > 0$ then B is infinite so $B \in p_j$ for some j and hence $B \cap X \neq \emptyset$. This proves $\nu_n^*(X) = 1$.

Ad (d). By (a) every $B \in \mathfrak{B}$ that is contained in X is in fact contained in some finite union of A_j 's; hence

$$(\nu_n)_*(X) = \sup_N \nu_n \left(\bigcup_{i \leq N} A_i \right).$$

Observe that by condition (3), if $n \notin X$ and $k < N$ then

$$(6.3.1) \quad \nu_n \left(\bigcup_{i=k}^N A_i \right) \leq \sum_{i=k}^N \nu_n(A_i) \leq \sum_{i=k}^N 2^{-(i+2)} < 2^{-(k+1)}.$$

Using the above estimate we can compute the limit of inner measures:

$$\begin{aligned} \lim_{n \notin X} \nu_{n*}(X) &= \lim_{n \notin X} \sup_N \nu_n \left(\bigcup_{i \leq N} A_i \right) \\ &\leq \lim_{n \notin X} \sup_N \left(\nu_n \left(\bigcup_{i < k} A_i \right) + \nu_n \left(\bigcup_{i=k}^N A_i \right) \right) \\ &\leq \lim_{n \notin X} \left(\nu_n \left(\bigcup_{i < k} A_i \right) + 2^{-(k+1)} \right) \text{ (by (6.3.1))} \\ &= \lim_{n \notin X} \nu_n \left(\bigcup_{i < k} A_i \right) + 2^{-(k+1)}. \end{aligned}$$

But for any $k \in \omega$ the set $\bigcup_{i < k} A_i \in \mathfrak{B}$ so we have $\lim_{n \notin X} \nu_n(\bigcup_{i < k} A_i) = 0$. Since k is arbitrary, this proves (d).

Once we know that X satisfies (a)-(d), we can check that X is indeed the set are looking for. Let, for every j , $\tilde{p}_j \in \text{ult}(\mathfrak{B}[X])$ be an arbitrary extension of p_j . Because $X \supseteq A_j$ and $A_j \in p_j$ so $X \in \tilde{p}_j$. Thus $\omega \setminus X$ omits all the ultrafilters \tilde{p}_j and, since it is an infinite set,

$$\widehat{\omega \setminus X} \cap K_{\mathfrak{B}[X]}^* \neq \emptyset,$$

which indicates that \tilde{p}_j are not dense in $K_{\mathfrak{B}[X]}^*$.

Now appealing to Theorem 3.2 we define the measures $\tilde{\nu}_n$ on $\mathfrak{B}[X]$ extending ν_n so that

$$(6.3.2) \quad \tilde{\nu}_n(X) = \begin{cases} \nu_n^*(X) & \text{for } n \in X, \\ (\nu_n)_*(X) & \text{for } n \notin X. \end{cases}$$

Then for $n \in X$ we have $\tilde{\nu}_n(X) = \nu_n^*(X) \rightarrow 1$ and for $n \notin X$ we have $\tilde{\nu}_n(X) = (\nu_n)_*(X) \rightarrow 0$ by property (d). Using Lemma 3.3 we conclude that $\tilde{\nu}_n - \delta_n \rightarrow 0$ on $\mathfrak{B}[X]$, and the proof is complete. \square

We have already all essential ingredients to carry out a diagonal construction leading to Theorem 6.1.

Proof of Theorem 6.1. Let \mathfrak{A}_0 be the algebra from Lemma 6.2 and let $(\nu_n^0)_n$ be a sequence of nonatomic probability measures on \mathfrak{A}_0 such that $\nu_n^0 - \delta_n \rightarrow 0$.

We construct inductively, for $\xi < \omega_1$, a sequence of countable algebras $\mathfrak{A}_\xi \subseteq P(\omega)$, sets $X_\xi \subseteq \omega$ and sequences $(\nu_n^\xi)_n$ of probability measures on \mathfrak{A}_ξ such that

- (i) $\mathfrak{A}_\beta \subseteq \mathfrak{A}_\xi$ for all $\beta < \xi < \omega_1$;
- (ii) \mathfrak{A}_ξ is generated by $\bigcup_{\beta < \xi} \mathfrak{A}_\beta$ and X_ξ ;
- (iii) $\nu_n^\xi|_{\mathfrak{A}_\beta} = \nu_n^\beta$ for every n and $\beta < \xi$;
- (iv) $\nu_n^\xi - \delta_n \rightarrow 0$ on \mathfrak{A}_ξ for every ξ .

Then we consider the algebra $\mathfrak{A} = \bigcup_{\xi < \omega_1} \mathfrak{A}_\xi$; for every n let μ_n be the unique probability measure on \mathfrak{A} such that $\mu_n|_{\mathfrak{A}_\xi} = \nu_n^\xi$ for $\beta < \xi$. It is clear that $\mu_n - \delta_n \rightarrow 0$ on \mathfrak{A} so $K_{\mathfrak{A}}$ is a smooth compactification of ω by Lemma 3.1. Therefore it is enough to check that by a suitable choice of sets X_ξ , we can guarantee that $K_{\mathfrak{A}}^*$ is not separable.

Using CH we fix an enumeration $\{D(\xi) : \alpha < \omega_1\}$ of all countable dense sets in $K_{\mathfrak{A}_0}^*$. At step ξ we let $\mathfrak{B} = \bigcup_{\alpha < \xi} \mathfrak{A}_\alpha$ and consider a sequence of measures ν_n defined on \mathfrak{B} , where ν_n is the unique extensions of ν_n^α , $\alpha < \xi$. Then we apply Lemma 6.3 to find a set X_ξ such that (iv) is granted for $\mathfrak{A}_\xi = \mathfrak{B}[X_\xi]$ and, at the same time X_ξ witnesses any extensions of ultrafilters from $D(\xi)$ are no longer dense in $K_{\mathfrak{A}_\xi}^*$.

It follows $K_{\mathfrak{A}}^*$ is not separable. Indeed, if we had a countable dense set $D \subseteq K_{\mathfrak{A}}^*$ then the set $D = \{x|_{\mathfrak{A}_0} : x \in X\}$ would be dense in $K_{\mathfrak{A}_0}^*$. But $D = D_\xi$ for some $\xi < \omega_1$ and D_ξ is not dense in \mathfrak{A}_ξ . \square

7. SMALL AND UGLY

We construct in this section a relatively small compactification $\gamma\omega$ which is not smooth, contrastive with the compactification from section 6.

Theorem 7.1. *There exists a non-smooth compactification $\gamma\omega$ which is first-countable and which remainder $\gamma\omega \setminus \omega$ is separable.*

We again construct a certain algebra $\mathfrak{A} \subseteq P(\omega)$; this time the main idea is to keep a fixed countable dense set of ultrafilter and to kill all possible sequences of measures that would witness the smoothness.

We shall use the notion of minimal extensions of algebras introduced by Koppelberg [19] which we recall now in the context of subalgebras of $P(\omega)$. The basic facts we list below can be found in [19] or [4].

If $\mathfrak{A} \subseteq P(\omega)$ and $X \subseteq \omega$ then $\mathfrak{A}[X]$ is said to be a *minimal extension* of \mathfrak{A} if for any algebra \mathfrak{B} , if $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{A}[X]$ then either $\mathfrak{B} = \mathfrak{A}$ or $\mathfrak{B} = \mathfrak{A}[X]$. This is equivalent to saying that for every $A \in \mathfrak{A}$, either $X \cap A \in \mathfrak{A}$ or $X \setminus A \in \mathfrak{A}$.

If $\mathfrak{A}[X] \neq \mathfrak{A}$ is a minimal extension then there is exactly one $q(X) \in \text{ult}(\mathfrak{A})$ that gets split in $\mathfrak{A}[X]$; this is $q(X) = \{A \in \mathfrak{A} : A \cap X \notin \mathfrak{A}\}$. Then every ultrafilter $p \neq q(X)$ has a unique extension to $\tilde{p} \in \text{ult}(\mathfrak{A}[X])$.

Given a sequence $A_n \in \mathfrak{A}$ and $p \in \text{ult}(\mathfrak{A})$, it will be convenient to say that A_n converge to p if every $B \in p$ contains almost all A_n .

Lemma 7.2. *If $\mathfrak{A}[X]$ is a minimal extension of \mathfrak{A} and $\mu \in \text{ba}_+(\mathfrak{A})$ does not have an atom at $q(X)$ then μ has a unique extension to $\tilde{\mu} \in \text{ba}_+(\mathfrak{A}[X])$.*

Proof. For every $\varepsilon > 0$ there is $A \in q(X)$ with $\mu(A) < \varepsilon$. Then $B = X \setminus A \in \mathfrak{A}$ so we have $B \subseteq X \subseteq B_1 = B \cup A$, where $B, B_1 \in \mathfrak{A}$ and $\mu(B_1 \setminus B) < \varepsilon$. This shows that $\mu_*(X) = \mu^*(X)$ must be equal to $\tilde{\mu}(X)$, whenever $\tilde{\mu}$ is a nonnegative extension of μ . We can repeat this argument with $A \cap X$ and $A \setminus X$ for $A \in \mathfrak{A}$ to conclude that $\tilde{\mu} = \mu^* = \mu_*$ on $\mathfrak{A}[X]$. \square

Lemma 7.3. *Let $\mathfrak{B} \subseteq P(\omega)$ be a algebra containing fin with \mathfrak{B}/fin nonatomic. Let $C = \{p_j : j \in \omega\}$ be a dense subset of $K_{\mathfrak{B}}^*$. Further let*

- (i) $q \in K_{\mathfrak{B}}^* \setminus C$ be a point of countable character;
- (ii) $(n_k)_k$ be a sequence on ω such that $n_k \rightarrow q$ (in the space $K_{\mathfrak{B}}$);
- (iii) $(B_k)_k$ and $(D_k)_k$ be sequences in \mathfrak{B} of infinite sets that converge to q and such that $B_k \cap D_j = \emptyset$ for all j, k .

If we let

$$X = \{n_k : k \in \omega\} \cup \bigcup_k B_k,$$

then $\mathfrak{B}[X]$ is a minimal extension of \mathfrak{B} and only $q = q(X)$ may be split in $\mathfrak{B}[X]$. Consequently, every p_j has a unique extension to an ultrafilter \tilde{p}_j on $\mathfrak{B}[X]$ and $\tilde{C} = \{\tilde{p}_j : j \in \omega\}$ is dense in $K_{\mathfrak{B}[X]}^*$.

Proof. For every $B \in q$, we have $X \setminus B \in \mathfrak{B}$ since $B_k \cup \{n_k\}$ converge to q . Hence $\mathfrak{B}[X]$ is a minimal extension.

To complete the proof we have to check the density of \tilde{C} , i.e. that for every $B \in \mathfrak{B}$, if $B \cap X$ is infinite then $B \cap X \in \tilde{p}_j$ for some j , and if $B \setminus X$ is infinite then $B \setminus X \in \tilde{p}_j$ for some j (for any $B \in \mathfrak{B}$).

Take $B \in \mathfrak{B}$ such that $B \cap X$ is infinite. If $B \in q$ then $B_k \subseteq B \cap X$ for some k and, taking j with $B_k \in p_j$, we get $B \cap X \in \tilde{p}_j$. If $B \notin q$ then $B \cap X \in \mathfrak{B}$ (by (ii) and (iii)) so $B \cap X \in p_j$ for some j .

Suppose now that $B \setminus X$ is infinite. If $B \notin q$ then $B \setminus X \in \mathfrak{B}$, as above. Finally, if $B \in q$ then there is k such that $D_k \subseteq B$. Then $D_k \subseteq B \setminus X$ by (iii) and, taking j with $D_k \in p_j$, we get $B \setminus X \in \tilde{p}_j$, so the proof is complete. \square

Here comes the lemma which constitutes the essence of the inductive step.

Lemma 7.4. *Let $\mathfrak{B} \subseteq P(\omega)$ be an algebra such that \mathfrak{B}/fin is nonatomic and $K_{\mathfrak{B}}^*$ is first-countable. Let $(p_j)_j$ be a sequence of nonprincipal ultrafilters on \mathfrak{B} such that the set $C = \{p_j : j < \omega\}$ is dense in $K_{\mathfrak{B}}^*$. Suppose also that $(\nu_n)_n$ is a sequence of measures on \mathfrak{B} such that $\nu_n|_{\text{fin}} \equiv 0$ for every n and $\nu_n - \delta_n \rightarrow 0$ on \mathfrak{B} .*

Then there exists a set $X \subseteq \omega$ such that

- (1) every p_j has a unique extension to $\tilde{p}_j \in \text{ult}(\mathfrak{B}[X])$ and the set $\{\tilde{p}_j : j < \omega\}$ is dense in $K_{\mathfrak{B}[X]}^*$;
- (2) if $\bar{\nu}_n$ is an extension of ν_n to a measure on $\mathfrak{B}[X]$ and $\|\bar{\nu}_n\| = \|\nu_n\|$ for every n then $\bar{\nu}_n(X) - \delta_n(X) \not\rightarrow 0$.

Proof. We shall choose an ultrafilter q on \mathfrak{B} and construct $B_n, D_n \in \mathfrak{B}$ and numbers n_k as in Lemma 7.1 so that the set $X = \{n_1, n_2, \dots\} \cup \bigcup_n B_n$ satisfies condition (2). Then (1) is granted by Lemma 7.3.

Take a point q in $K_{\mathfrak{B}}^* \setminus C$ which is an atom of no measure ν_n . Let $\{U_k : k \in \omega\} \subseteq \mathfrak{B}$ be a base at $q \in K_{\mathfrak{B}}$. Choose also a sequence $(m_i)_i$ in ω such that $m_i \rightarrow q$ in the space $K_{\mathfrak{B}}$ and put $N = \{m_i : i \in \omega\}$.

We construct inductively $C_k, D_k, V_k \in \mathfrak{B}$ and $n_k \in N$ such that for every k

- (a) $V_k \setminus (B_k \cup D_k) \in q$ and $B_k \cup D_k \subseteq V_k \subseteq U_k$;
- (b) $n_k \in V_{k-1} \supseteq V_k$;
- (c) $B_i \cap D_j = \emptyset$ for all i, j ;
- (d) $\nu_{n_k} \left(\bigcup_{j < k} B_j \right) < 1/k$;
- (e) $|\nu_{n_k}|(V_k) < 1/k$.

The inductive construction is straightforward: if we carried it out below k then set

$$B = \bigcup_{j \leq k-1} B_j, \quad D = \bigcup_{j \leq k-1} D_j, \quad V = V_{k-1} \cap U_k \setminus (B \cup D).$$

Then $V \in q$ so V contains infinitely many m_i (as $m_i \rightarrow q$) and among them we choose n_k so that (d) holds (using $\nu_n - \delta_n \rightarrow 0$). Then we choose $V_k \subseteq V$ satisfying (e) (as q is not an atom of ν_{n_k}). Finally, we can choose $B_k, D_k \subseteq V_k \setminus (B \cup D)$ so that (a), (b) and (c) hold.

Recall that $X = \{n_1, n_2, \dots\} \cup \bigcup_n B_n$. Consider now any extensions of measures ν_n to $\bar{\nu}_n \in \text{ba}(\mathfrak{B}[X])$ with $\|\bar{\nu}_n\| = \|\nu_n\|$. Note that in such a case $|\bar{\nu}_n|$ is an extension of $|\nu_n|$. Since we have for any k (using (a))

$$\bigcup_{n \in \omega} B_n = \bigcup_{j < k} B_j \cup \bigcup_{j \geq k} B_j \subseteq \bigcup_{j < k} B_j \cup V_k,$$

and the set $N \setminus V_k$ is finite, we get

$$\bar{\nu}_{n_k}(X) \leq \nu_{n_k} \left(\bigcup_{j < k} B_j \right) + |\bar{\nu}_{n_k}|(V_k) < 1/k + |\bar{\nu}_{n_k}|(V_k) = 1/k + |\nu_{n_k}|(V_k) < 2/k.$$

But $n_k \in X$, so $\nu_n(X) - \delta_n(X) \not\rightarrow 0$, and we are done. \square

Proof of Theorem 7.1. We first describe a certain operation that will be iterated to construct an algebra defining the required compactification.

Consider an algebra \mathfrak{B} in $P(\omega)$ containing fin and such that \mathfrak{B}/fin is nonatomic. Suppose also that

- (1) $K_{\mathfrak{B}}$ is first-countable;
- (2) $C = \{p_j : j \in \omega\}$ is a dense subset of $K_{\mathfrak{A}_0}^*$;
- (3) $\text{ba}(\mathfrak{B})$ is of size \mathfrak{c} and $\nu^\xi = (\nu_n^\xi)_n$, $\xi < \mathfrak{c}$ is an enumeration of all bounded sequences of measures on \mathfrak{B}_ξ such that $\nu_n^\xi - \delta_n \rightarrow 0$ on \mathfrak{B}_ξ .

We fix a set $Q \subseteq K_{\mathfrak{B}}^* \setminus C$ of cardinality \mathfrak{c} and for every $\xi < \mathfrak{c}$ apply Lemma 7.4 to the sequence ν^ξ : pick $q_\xi \in Q \setminus \{q_\eta : \eta < \xi\}$ and form the set X_ξ as in Lemma 7.4.

Then we let $\mathfrak{B}^\#$ be the algebra generated by \mathfrak{B} and $\{X_\xi : \xi < \mathfrak{c}\}$. Note that

- (a) $K_{\mathfrak{B}^\#}$ is first-countable;
- (b) every p_j has a unique extension to $p_j^\# \in \text{ult}(\mathfrak{B}^\#)$;
- (c) $C^\# = \{p_j^\# : j \in \omega\}$ is dense in $K_{\mathfrak{B}^\#}^*$;
- (d) $|\text{ba}(\mathfrak{B}^\#)| = \mathfrak{c}$.

Ad (a). If $p \in \text{ult}(\mathfrak{B})$ is never split then it has a base in \mathfrak{B} . Otherwise $p = q_\xi$ for some $\xi < \mathfrak{c}$ and it is split only by X_ξ into two ultrafilters having bases in $\mathfrak{B}[X_\xi]$.

Ad (b) and (c). This follows from Lemma 7.3

Ad (d). It suffices to note that any $\mu \in \text{ba}_+(\mathfrak{B})$ has at most \mathfrak{c} extensions to nonnegative measures on $\mathfrak{B}^\#$. Indeed, take $\mu \in \text{ba}_+(\mathfrak{B})$ and let $N \subseteq K_{\mathfrak{B}}^*$ be the set of all atoms of μ . Then N is countable and the algebra \mathfrak{B}' generated by \mathfrak{B} and $\{X_\xi : q_\xi \in N\}$ is countably generated over \mathfrak{B} . Therefore we can extend μ to a nonnegative measure μ' on \mathfrak{B}' in at most \mathfrak{c} ways. In turn every such μ' extends uniquely to a measure in $\text{ba}_+(\mathfrak{B}^\#)$ by Lemma 7.2.

We shall now iterate the operation $\#$. Let \mathfrak{A}_0 be a countable algebra in $P(\omega)$ such that $\mathfrak{A}_0/\text{fin}$ is nonatomic. Fix $C = \{p_j : j \in \omega\}$ as above and choose a pairwise disjoint family $\{Q_\alpha : \alpha < \omega_1\}$ such that $|Q_\alpha| = \mathfrak{c}$ and $Q_\alpha \subseteq K_{\mathfrak{A}_0}^* \setminus C$ for every $\alpha < \omega_1$.

We define $\mathfrak{A}_{\alpha+1} = (\mathfrak{A}_\alpha)^\#$, with Q_α playing the role of Q in the construction. We also let $\mathfrak{A}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{A}_\beta$ for $\alpha < \omega_1$ limit, and claim that $\mathfrak{A} = \bigcup_{\alpha < \omega_1} \mathfrak{A}_\alpha$ is the required algebra.

The compactification $K_{\mathfrak{A}}$ is not smooth by Lemma 3.1. Indeed, take any $\mu_n \in \text{ba}(\mathfrak{A})$ with $\sup_n \|\mu_n\| < \infty$. Note that every measure on \mathfrak{A} attains its norm on some countable subalgebra. Hence there is $\alpha < \omega_1$ such that, writing $\nu_n = \mu_n|_{\mathfrak{A}_\alpha}$, we have $\|\nu_n\| = \|\mu_n\|$ for every n . Then $(\nu_n)_n = \nu^\xi$ for some $\xi < \mathfrak{c}$ and by our construction $\nu_n(X_\xi) - \delta_n(X_\xi) \not\rightarrow 0$.

Finally, $K_{\mathfrak{A}}^*$ is separable because C remains dense throughout the construction. The fact that $K_{\mathfrak{A}}$ is first-countable follows from the fact that Q_α 's are pairwise disjoint, as in the proof of (a) above. \square

In the terminology of [19], if $\mathfrak{B} \subseteq \mathfrak{A} \subseteq P(\omega)$ then \mathfrak{A} is minimally generated over \mathfrak{B} if, for some ξ , \mathfrak{A} is a continuous increasing union $\mathfrak{A} = \bigcup_{\alpha < \xi} \mathfrak{A}_\alpha$, where $\mathfrak{A}_0 = \mathfrak{B}$ and $\mathfrak{A}_{\alpha+1}$ is a minimal extension of \mathfrak{A}_α for every $\alpha < \xi$. An algebra \mathfrak{A} is minimally generated if it is minimally generated over the trivial algebra. It is clear that the algebra \mathfrak{A} we constructed in the proof of Theorem 7.1 is minimally generated.

In the language of extension operators, Theorem 7.1 says the following.

Corollary 7.5. *There exist a separable first-countable compact space L and a compact superspace K with $K \setminus L$ countable such that there is no extension operator $C(L) \rightarrow C(K)$.*

Proof. Take $\gamma\omega$ from Theorem 7.1, put $L = \gamma\omega \setminus \omega$, $K = \gamma\omega$ and apply Lemma 2.4. \square

8. ON HEREDITARILY SOBCZYK SPACES

In this final section we prove a general result on compacta K for which the space $C(K)$ contains a rich family of complemented copies of c_0 . In the definition below we use the terminology from Džamonja and Kunen [13].

Definition 8.1. *We say that a compact space K is in the class (MS) (of measure separable spaces) if every probability Radon measure μ on K has the countable Maharam type, i.e. $L_1(\mu)$ is a separable Banach space.*

Recall also the following standard notion.

Definition 8.2. *Let X be any vector space and let $(x_n)_n$ be a sequence in X . A sequence $(y_n)_n$ is a convex block subsequence of $(x_n)_n$ if there exist finite sets $I_n \subset \omega$ with $\max I_n < \min I_{n+1}$, and a function $a : \omega \rightarrow \mathbb{R}_+$ such that for all $n \in \omega$*

$$y_n = \sum_{j \in I_n} a(j) x_j \quad \text{and} \quad \sum_{j \in I_n} a(j) = 1.$$

Our result is a consequence of (a particular case of) a result due to Haydon, Levy and Odell [18], see also Krupski and Plebanek [20] for a direct approach to the following.

Theorem 8.3 (Haydon, Levy, Odell). *If K is a compact space in the class (MS) then every bounded sequence $(\mu_n)_n$ in $\mathcal{M}(K)$ has a convex block subsequence $(\nu_n)_n$ converging to some measure $\nu \in \mathcal{M}(K)$.*

Theorem 8.4. *Let K be a compact space in (MS). Then for any isomorphic embedding $T : c_0 \rightarrow C(K)$ the space $T[c_0]$ contains a subspace Y which is isomorphic to c_0 and complemented in $C(K)$,*

Proof. Let (e_n) be the sequence of unit vectors in c_0 ; we write $e_n^* \in c_0^* = l_1$.

Given an isomorphic embedding $T : c_0 \rightarrow C(K)$, put $g_n = Te_n$ for every n . Since T is an embedding, the dual operator

$$T^* : C(K)^* = \mathcal{M}(K) \rightarrow c_0^* = l_1,$$

is onto and therefore there is a bounded sequence of measures $\mu_n \in \mathcal{M}(K)$ such that $T^*\mu_n = e_n^*$. Then we have $\mu_n(g_k) = \mu_n(Te_k) = T^*\mu_n(e_k) = e_n^*(e_k)$.

Consider the sequence of measures $(\mu_n)_n$. By Theorem 8.3 it has a convex block subsequence $(\nu_n)_n$ converging to some measure $\nu \in \mathcal{M}(K)$. Say that $\nu_n = \sum_{k \in I_n} a(k)\mu_k$, where I_n and a are as in Definition 8.2.

For every n put $\bar{e}_n = \sum_{k \in I_n} e_k$. Then \bar{e}_n are norm-one vectors spanning a subspace X of c_0 that is clearly isometric to c_0 . Hence the functions $h_n = T\bar{e}_n \in C(K)$ span a subspace

$Y = T[X]$ of $T[c_0]$ that is isomorphic to c_0 and it is enough to check that Y is complemented in $C(K)$.

Since $T^*\mu_n = e_n^*$, we have $T^*\nu_n = \sum_{i \in I_n} t(i)e_i^*$, and

$$\nu_n(h_k) = \nu_n(T\bar{e}_k) = T^*\nu_n(\bar{e}_k) = \sum_{i \in I_n} t(i)e_i^* \left(\sum_{j \in I_k} e_j \right) = \sum_{i \in I_n, j \in I_k} t(i)e_i^*(e_j),$$

which is equal to 0 if $n \neq k$ (since then $I_n \cap I_k = \emptyset$), and is equal to $\sum_{i \in I_n} t(i) = 1$ when $n = k$.

Now, as in Lemma 2.1, we conclude that $P : C(K) \rightarrow C(K)$ defined by

$$Pf = \sum_{n \in \omega} (\nu_n - \nu)(f) \cdot h_n,$$

is a bounded projection onto Y . Indeed,

$$\nu(h_n) = \lim_j \nu_j(h_n) = \lim_j T^*\nu_j(\bar{e}_n) = \lim_j \left(\sum_{i \in I_j} t(i)e_i^* \right) (\bar{e}_n) = 0,$$

for every n . This shows that $Ph_n = h_n$; moreover, $Pf \in Y$ for any $f \in C(K)$ since since $\nu_n(f) - \nu(f) \rightarrow 0$ for every n . \square

A Banach space X having the property that every isomorphic copy of c_0 in X has a subspace isomorphic to c_0 and complemented in X is called *hereditarily separably Sobczyk* in [16]. Theorem 8.4 states that $C(K)$ is such a space whenever K is in the class (MS). Moltó [22] gave an example of a Rosenthal compact space K such that $C(K)$ does not have the Sobczyk property. Rosenthal compacta are in (MS), due to a results of Bourgain and Todorčević, see [12] for a more general result and references therein. Consequently, $C(K)$ is hereditarily separably Sobczyk whenever K is Rosenthal compact.

The final result is related to our Theorem 7.1.

Corollary 8.5. *If \mathfrak{A} is a minimally generated Boolean algebra then $C(K_{\mathfrak{A}})$ is hereditarily separably Sobczyk.*

Proof. This follows from Theorem 8.4 and the remark after it, and a result due to Borodulin-Nadzieja [4] stating that $K_{\mathfrak{A}}$ is in the class (MS) whenever \mathfrak{A} is minimally generated. \square

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